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Equations of the spherical conics

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Abstract

If we transform definitions of the conics in Euclidean plane on sphere, we obtain spherical conics. Although many equations related to spherical conics have been known, those equations depend on many parameters. In this study, we have work about the method that we have developed for equations which depend on one parameter of the spherical conics.

1. Introduction

We know that spherical ellipse is the intersection of the unit sphere with a quadratic cone whose vertex is at the center of the sphere [2]. Spherical ellipse equation that we had by these means depend on more than one parameter. We have wondered whether we could write spherical ellipse equation depending on one parameter or not, then we developed the method below:

Take two fixed points F_1 and F_2 on Euclidean plane, then the locus of the points X such that

$$d(X,F_1)+d(X,F_2)=2b, 0<b\in R$$

is called a *planar ellipse*.

Secondly, given two points F_1 and F_2 on unit sphere $S^2\subset R^3$, let θ be the angle subtended at the center of the S^2, $d(F_1,F_2)$ denote the *geodesic distance* between F_1 and F_2. If we apply planar ellipse definition on S^2, then we may define spherical ellipse, that is

Take two fixed points F_1 and F_2 on S^2, then the locus of the points X such that

$$d(X,F_1)+d(X,F_2)=2b, 0<b\in R$$

is called a *spherical ellipse*.

If we use spherical ellipse definition on S^2, we obtain equations depending on more than one parameter [2]. We can reduce the spherical ellipse equations to one parameter by using the basis that we extended by F_1 and F_2. Similarly we had equations of other conics, such as *spherical hyperbola* and *spherical parabola* depending on one parameter.

2. Spherical ellipse

Take two fixed points F_1 and F_2 on S^2 with the properties of $d(F_1, F_2) = 2c, 0 < c < b \in R$ and the set of points

$$S = \{X \mid d(X, F_1) + d(X, F_2) = 2b, X \in S^2\}$$

If we define $\overrightarrow{F_3} = \frac{\overrightarrow{F_1} \times \overrightarrow{F_2}}{\|\overrightarrow{F_1} \times \overrightarrow{F_2}\|}$, then $\overrightarrow{F_1}, \overrightarrow{F_2}$ and $\overrightarrow{F_3}$ will be linearly independent.

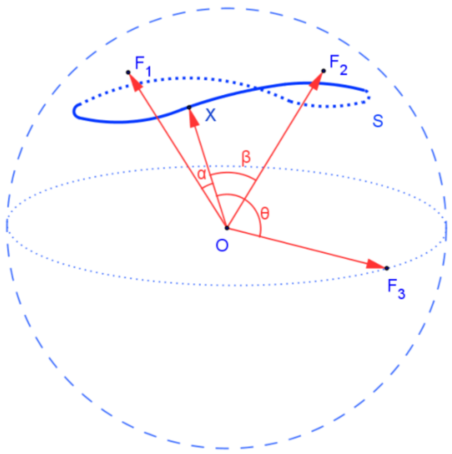

Figure 1: Spherical ellipse

Hence, according to the Figure 1, we have

$$d(X, F_1) = \alpha, d(X, F_2) = \beta, d(X, F_3) = \theta$$
$$\alpha + \beta = 2b$$

For any point X on S², we can write

$$\vec{X} = \lambda_1 \overrightarrow{F_1} + \lambda_2 \overrightarrow{F_2} + \lambda_3 \overrightarrow{F_3}, \lambda_i \in \mathrm{R}, \mathrm{i} = 1, 2, 3 \qquad (1)$$

Thus, we can get the equations

$$\vec{X} \cdot \overrightarrow{F_1} = \lambda_1 + \lambda_2 \overrightarrow{F_1} \cdot \overrightarrow{F_2} = \cos\alpha$$
$$\vec{X} \cdot \overrightarrow{F_2} = \lambda_1 \overrightarrow{F_1} \cdot \overrightarrow{F_2} + \lambda_2 = \cos\beta$$
$$\vec{X} \cdot \overrightarrow{F_3} = \lambda_3 = \cos\theta$$

From these equations simply, we obtain following facts,

$$\lambda_1 + \lambda_2 \cos 2c = \cos\alpha$$
$$\lambda_1 \cos 2c + \lambda_2 = \cos\beta$$
$$\lambda_3 = \cos\theta$$

By solving these equations we obtain easily λ_1, λ_2, and λ_3. That is

$$\lambda_1 = \frac{\begin{vmatrix} \cos\alpha & \cos 2c \\ \cos\beta & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \cos 2c \\ \cos 2c & 1 \end{vmatrix}} = \frac{\cos\alpha - \cos(2b - \alpha)\cos 2c}{\sin^2 2c}$$

$$\lambda_2 = \frac{\begin{vmatrix} 1 & \cos\alpha \\ \cos 2c & \cos\beta \end{vmatrix}}{\begin{vmatrix} 1 & \cos 2c \\ \cos 2c & 1 \end{vmatrix}} = \frac{\cos(2b-\alpha) - \cos\alpha\cos 2c}{\sin^2 2c}$$

$$\lambda_3 = \cos\theta$$

Hence, for any point X on S^2, we can write (1) as follows:

$$\vec{X} = \frac{\cos\alpha - \cos(2b-\alpha)\cos 2c}{\sin^2 2c}\overrightarrow{F_1} + \frac{\cos(2b-\alpha) - \cos\alpha\cos 2c}{\sin^2 2c}\overrightarrow{F_2} + \cos\theta\,\overrightarrow{F_3} \tag{2}$$

and we know

$$\vec{X} = (x_1, x_2, x_3),\ \left\|\vec{X}\right\|^2 = 1,\ X \in S^2 \tag{3}$$

After combining (2) and (3), we have $\vec{X}$ depending on one parameter, that is shown below in Example 1.

Example 1: Let us take $\overrightarrow{F_1} = (1,0,0)$ and $\overrightarrow{F_2} = (0,1,0)$ then $\overrightarrow{F_3} = \frac{\overrightarrow{F_1}\times\overrightarrow{F_2}}{\left\|\overrightarrow{F_1}\times\overrightarrow{F_2}\right\|} = (0,0,1)$. If we plug in $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, and $\overrightarrow{F_3}$ in equation (2), we have equation (4), which is

$$\vec{X} = \left(\frac{\cos\alpha - \cos(2b-\alpha)\cos 2c}{\sin^2 2c}, \frac{\cos(2b-\alpha) - \cos\alpha\cos 2c}{\sin^2 2c}, \cos\theta\right) \tag{4}$$

Clearly $d(F_1, F_2) = 2c$,

$$2c = \frac{\pi}{2}$$

and we know

$$\vec{X} = (x_1, x_2, x_3),\ \left\|\vec{X}\right\|^2 = 1,\ X \in S^2$$

Using these facts in equation (4), simply we have

$$\left(\frac{\cos\alpha - \cos(2b-\alpha)\cos 2c}{\sin^2 2c}\right)^2 + \left(\frac{\cos(2b-\alpha) - \cos\alpha\cos 2c}{\sin^2 2c}\right)^2 + \cos^2\theta = 1$$

$$\left(\frac{\cos\alpha - \cos(2b-\alpha).0}{1}\right)^2 + \left(\frac{\cos(2b-\alpha) - \cos\alpha.0}{1}\right)^2 + \cos^2\theta = 1$$

$$1 - \cos^2\alpha - \cos^2(2b-\alpha) = \cos^2\theta$$

So that

$$\vec{X} = \left(\cos\alpha, \cos(2b-\alpha), \pm\sqrt{1 - \cos^2\alpha - \cos^2(2b-\alpha)}\right)$$

Hence, we can get the following parametric equations

$$x_1 = \cos\alpha$$
$$x_2 = \cos(2b-\alpha)$$
$$x_3 = \pm\sqrt{1 - \cos^2\alpha - \cos^2(2b-\alpha)}$$

Now, let's take $2b = \frac{2\pi}{3}$. As a result, we will have the following equations

$$x_1 = \cos\alpha$$
$$x_2 = \cos\left(\frac{2\pi}{3} - \alpha\right)$$
$$x_3 = \pm\sqrt{1 - \cos^2\alpha - \cos^2\left(\frac{2\pi}{3} - \alpha\right)}$$

In these three equations, for $\alpha \in [0,2\pi)$ we will have the following graphs.

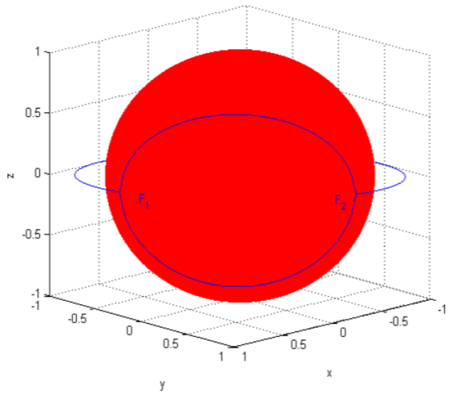

Figure 2: Spherical ellipse (Plots produced with Matlab by using the command below)

```
hold on;
ezplot3('cos(t)','cos(2*pi/3-t)','sqrt(1-(cos(t)).^2-(cos(2*pi/3-t)).^2)'),
ezplot3('cos(t)','cos(2*pi/3-t)','-sqrt(1-(cos(t)).^2-(cos(2*pi/3-t)).^2)'),
sphere(100),text(1,0,0,'.F_1','FontSize',10),text(0,1,0,'.F_2','FontSize',10)
```

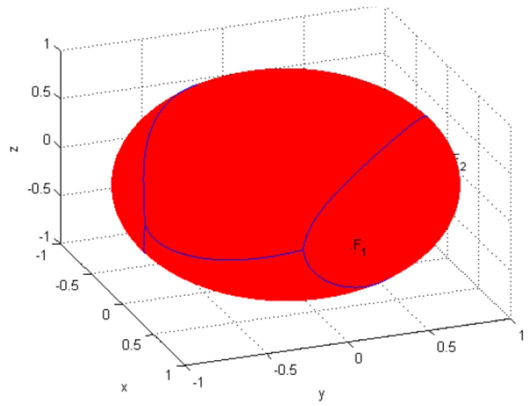

Figure 3: Spherical ellipse

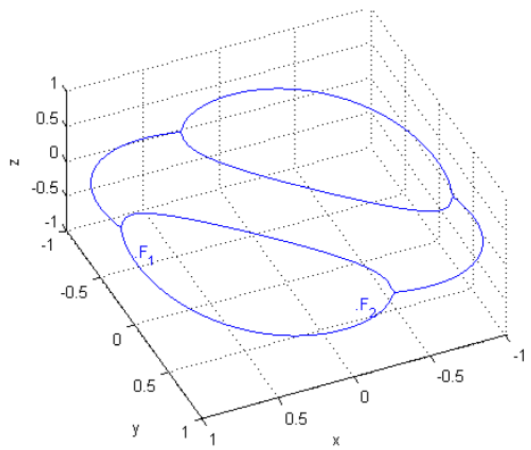

Figure 4: Spherical ellipse (without sphere)

For $\alpha \in [\frac{\pi}{12}, \frac{7\pi}{12})$ we will have the following graphs.

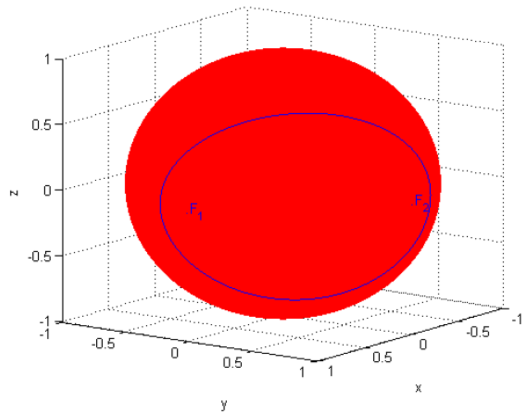

Figure 5: Spherical ellipse (Plots produced with Matlab by using the command below)

```
hold on;
ezplot3('cos(t)','cos(2*pi/3-t)','sqrt(1-(cos(t)).^2-(cos(2*pi/3-t)).^2)',[pi/12,(7*pi)/12]),
ezplot3('cos(t)','cos(2*pi/3-t)','-sqrt(1-(cos(t)).^2-(cos(2*pi/3-t)).^2)',[pi/12,(7*pi)/12]),
sphere(100),text(1,0,0,'.F_1','FontSize',10),text(0,1,0,'.F_2','FontSize',10)
```

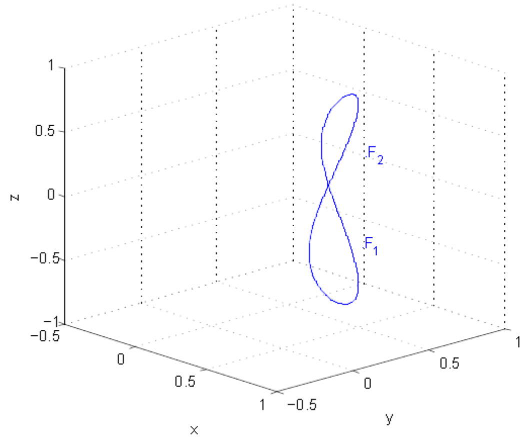

Figure 6: Spherical ellipse (without sphere)

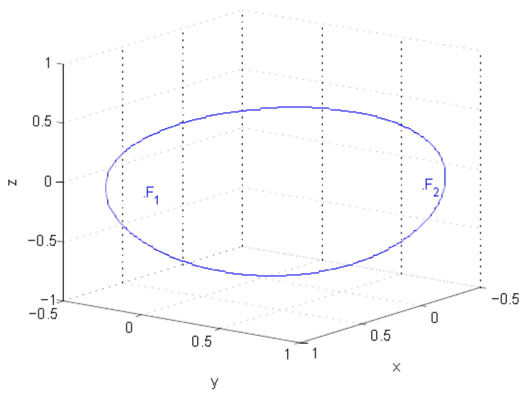

Figure 7: Spherical ellipse

3. Spherical hyperbola

Take two fixed points F_1 and F_2 on S^2 with the properties of $d(F_1, F_2) = 2c, 0 < b < c \in R$ and the set of points

$$S = \{X \mid |d(X, F_1) - d(X, F_2)| = 2b, X \in S^2\}$$

If we define $\overrightarrow{F_3} = \frac{\overrightarrow{F_1} \times \overrightarrow{F_2}}{\|\overrightarrow{F_1} \times \overrightarrow{F_2}\|}$, then $\overrightarrow{F_1}, \overrightarrow{F_2}$, and $\overrightarrow{F_3}$ will be linearly independent.

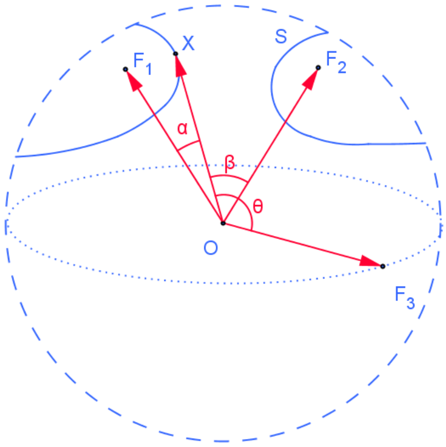

Figure 8: Spherical hyperbola

Hence, according to the Figure 8, we have

$$d(X, F_1) = \alpha, d(X, F_2) = \beta, d(X, F_3) = \theta$$
$$|\alpha - \beta| = 2b$$

For any point X on S^2, we can write

$$\vec{X} = \lambda_1 \overrightarrow{F_1} + \lambda_2 \overrightarrow{F_2} + \lambda_3 \overrightarrow{F_3}, \lambda_i \in \mathrm{R}, \mathrm{i} = 1, 2, 3$$

Thus, we can get the equations

$$\vec{X} \cdot \overrightarrow{F_1} = \lambda_1 + \lambda_2 \overrightarrow{F_1} \cdot \overrightarrow{F_2} = \cos\alpha$$
$$\vec{X} \cdot \overrightarrow{F_2} = \lambda_1 \overrightarrow{F_1} \cdot \overrightarrow{F_2} + \lambda_2 = \cos\beta$$
$$\vec{X} \cdot \overrightarrow{F_3} = \lambda_3 = \cos\theta$$

From these equations simply, we obtain following facts,

$$\lambda_1 + \lambda_2 \cos 2c = \cos\alpha$$
$$\lambda_1 \cos 2c + \lambda_2 = \cos\beta$$
$$\lambda_3 = \cos\theta$$

By solving these equations, we obtain easily λ_1, λ_2, and λ_3. That is

$$\lambda_1 = \frac{\begin{vmatrix} \cos\alpha & \cos 2c \\ \cos\beta & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \cos 2c \\ \cos 2c & 1 \end{vmatrix}} = \frac{\cos\alpha - \cos\beta\cos 2c}{\sin^2 2c}$$

$$\lambda_2 = \frac{\begin{vmatrix} 1 & \cos\alpha \\ \cos 2c & \cos\beta \end{vmatrix}}{\begin{vmatrix} 1 & \cos 2c \\ \cos 2c & 1 \end{vmatrix}} = \frac{\cos\beta - \cos\alpha\cos 2c}{\sin^2 2c}$$

$$\lambda_3 = \cos\theta$$

As a result of

$$|\alpha - \beta| = 2b$$

we should examine two cases $\alpha > \beta$ and $\alpha < \beta$.

Case 1 ($\boldsymbol{\alpha > \beta}$):

$$\vec{X} = \frac{\cos\alpha - \cos(\alpha - 2b)\cos 2c}{\sin^2 2c}\overrightarrow{F_1} + \frac{\cos(\alpha - 2b) - \cos\alpha\cos 2c}{\sin^2 2c}\overrightarrow{F_2} + \cos\theta\,\overrightarrow{F_3} \tag{5}$$

Case 2 ($\boldsymbol{\alpha < \beta}$):

$$\vec{X} = \frac{\cos\alpha - \cos(\alpha + 2b)\cos 2c}{\sin^2 2c}\overrightarrow{F_1} + \frac{\cos(\alpha + 2b) - \cos\alpha\cos 2c}{\sin^2 2c}\overrightarrow{F_2} + \cos\theta\,\overrightarrow{F_3} \tag{6}$$

and we know

$$\vec{X} = (x_1, x_2, x_3),\ \|\vec{X}\|^2 = 1,\ X \in S^2$$

After combining (3) with (5) and (6), we have $\vec{X}$ depending on only one parameter, that is shown below in Example 2 and Example 3.

Example 2 ($\boldsymbol{\alpha > \beta}$) : Let us take $\overrightarrow{F_1} = (1,0,0)$ and $\overrightarrow{F_2} = (0,1,0)$ then $\overrightarrow{F_3} = \frac{\overrightarrow{F_1}\times\overrightarrow{F_2}}{\|\overrightarrow{F_1}\times\overrightarrow{F_2}\|} = (0,0,1)$. If we plug in $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, and $\overrightarrow{F_3}$ in equation (5), we have equation (7), which is

$$\vec{X} = \left(\frac{\cos\alpha - \cos(\alpha - 2b)\cos 2c}{\sin^2 2c}, \frac{\cos(\alpha - 2b) - \cos\alpha\cos 2c}{\sin^2 2c}, \cos\theta\right) \tag{7}$$

Clearly $d(F_1, F_2) = 2c$,

$$2c = \frac{\pi}{2}$$

and we know

$$\vec{X} = (x_1, x_2, x_3),\ \|\vec{X}\|^2 = 1,\ X \in S^2$$

Using these facts in equation (7), simply we have

$$\left(\frac{\cos\alpha - \cos(\alpha - 2b)\cos 2c}{\sin^2 2c}\right)^2 + \left(\frac{\cos(\alpha - 2b) - \cos\alpha\cos 2c}{\sin^2 2c}\right)^2 + \cos^2\theta = 1$$

$$\left(\frac{\cos\alpha - \cos(\alpha - 2b).0}{1}\right)^2 + \left(\frac{\cos(\alpha - 2b) - \cos\alpha.0}{1}\right)^2 + \cos^2\theta = 1$$

$$1 - \cos^2\alpha - \cos^2(2b - \alpha) = \cos^2\theta$$

So that

$$\vec{X} = \left(\cos\alpha, \cos(\alpha - 2b), \pm\sqrt{1 - \cos^2\alpha - \cos^2(\alpha - 2b)}\right)$$

We know that

$$\cos(\alpha - 2b) = \cos(2b - \alpha)$$

Hence, we can get the following parametric equations

$$x_1 = \cos\alpha$$
$$x_2 = \cos(2b - \alpha)$$
$$x_3 = \pm\sqrt{1 - \cos^2\alpha - \cos^2(2b - \alpha)}$$

In this case, spherical hyperbola and the spherical ellipse have the same equations.

Now, let's take $2b = \frac{\pi}{3}$. As a result, we will have the following equations.

$$x_1 = \cos\alpha$$
$$x_2 = \cos\left(\frac{\pi}{3} - \alpha\right)$$
$$x_3 = \pm\sqrt{1 - \cos^2\alpha - \cos^2\left(\frac{\pi}{3} - \alpha\right)}$$

In these three equations, for $\alpha \in [0,2\pi)$ we will have the following graph.

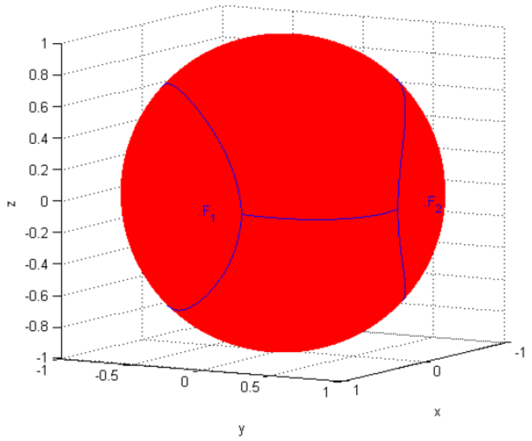

Figure 9: Spherical hyperbola (Plot produced with Matlab by using the command below)

```
hold on;
ezplot3('cos(t)','cos(pi/3-t)','sqrt(1-(cos(t)).^2-(cos(pi/3-t)).^2)'),
ezplot3('cos(t)','cos(pi/3-t)','-sqrt(1-(cos(t)).^2-(cos(pi/3-t)).^2)'),
sphere(100),text(1,0,0,'.F_1','FontSize',10),text(0,1,0,'.F_2','FontSize',10)
```

Example 3 $(\boldsymbol{\alpha < \beta})$ **:** Let us take $\overrightarrow{F_1} = (1,0,0)$ and $\overrightarrow{F_2} = (0,1,0)$ then $\overrightarrow{F_3} = \frac{\overrightarrow{F_1}\times\overrightarrow{F_2}}{\|\overrightarrow{F_1}\times\overrightarrow{F_2}\|} = (0,0,1)$. If we plug in $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, and $\overrightarrow{F_3}$ in equation (6), we have equation (8), which is

$$\vec{X} = \left(\frac{\cos\alpha - \cos(\alpha+2b)\cos 2c}{\sin^2 2c}, \frac{\cos(\alpha+2b) - \cos\alpha\cos 2c}{\sin^2 2c}, \cos\theta\right) \tag{8}$$

Clearly $d(F_1, F_2) = 2c$,

$$2c = \frac{\pi}{2}$$

and we know

$$\vec{X} = (x_1, x_2, x_3),\ \|\vec{X}\|^2 = 1, X \in S^2$$

Using these facts in equation (8), simply we have

$$\left(\frac{\cos\alpha - \cos(\alpha+2b)\cos 2c}{\sin^2 2c}\right)^2 + \left(\frac{\cos(\alpha+2b) - \cos\alpha\cos 2c}{\sin^2 2c}\right)^2 + \cos^2\theta = 1$$
$$\left(\frac{\cos\alpha - \cos(\alpha+2b)\,.0}{1}\right)^2 + \left(\frac{\cos(\alpha+2b) - \cos\alpha.0}{1}\right)^2 + \cos^2\theta = 1$$
$$1 - \cos^2\alpha - \cos^2(\alpha+2b) = \cos^2\theta$$

So that

$$\vec{X} = \left(\cos\alpha, \cos(\alpha + 2b), \pm\sqrt{1 - \cos^2\alpha - \cos^2(\alpha + 2b)}\right)$$

Hence, we can get the following parametric equations

$$x_1 = \cos\alpha$$
$$x_2 = \cos(\alpha + 2b)$$
$$x_3 = \pm\sqrt{1 - \cos^2\alpha - \cos^2(\alpha + 2b)}$$

Now, let's take $2b = \frac{\pi}{3}$. As a result, we will have the following equations.

$$x_1 = \cos\alpha$$
$$x_2 = \cos\left(\frac{\pi}{3} + \alpha\right)$$
$$x_3 = \pm\sqrt{1 - \cos^2\alpha - \cos^2\left(\frac{\pi}{3} + \alpha\right)}$$

In these three equations, for $\alpha \in [0,2\pi)$ we will have the following graphs.

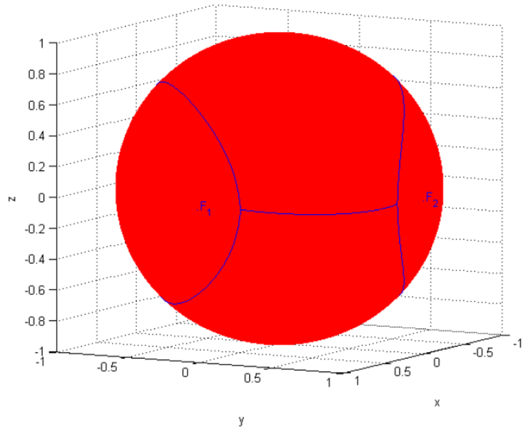

Figure 10: Spherical hyperbola (Plot produced with Matlab by using the command below)

```
hold on;
ezplot3('cos(t)','cos(pi/3+t)','sqrt(1-(cos(t)).^2-(cos(pi/3+t)).^2)'),
ezplot3('cos(t)','cos(pi/3+t)','-sqrt(1-(cos(t)).^2-(cos(pi/3+t)).^2)'),
sphere(100),text(1,0,0,'.F_1','FontSize',10),text(0,1,0,'.F_2','FontSize',10)
```

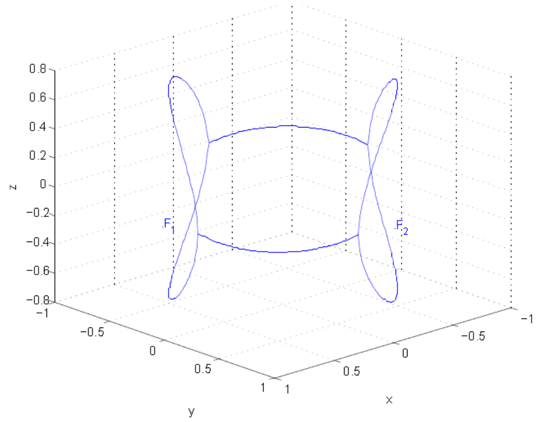

Figure 11: Spherical hyperbola (without sphere)

4. Spherical parabola

Take two fixed points F_1 and F_2 on S^2 with the properties of $d(F_1, F_2) = 2c, 0 < c \in R$ and S' be the great circle whose normal vector is $\overrightarrow{F_2}$. Let us take the set of points

$$S = \{X \mid d(X, F_1) + \frac{\pi}{2} = d(X, F_2)\,, X \in S^2\}$$

If we define $\overrightarrow{F_3} = \frac{\overrightarrow{F_1} \times \overrightarrow{F_2}}{\|\overrightarrow{F_1} \times \overrightarrow{F_2}\|}$, then $\overrightarrow{F_1}, \overrightarrow{F_2}$, and $\overrightarrow{F_3}$ will be linearly independent.

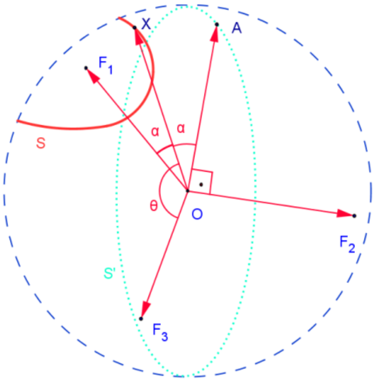

Figure 12: Spherical Parabola

Hence, according to the Figure 12, we have

$$d(X, F_1) = \alpha, d(X, F_2) = \frac{\pi}{2} + \alpha, d(X, F_3) = \theta$$

For any point X on S^2, we can write

$$\vec{X} = \lambda_1 \overrightarrow{F_1} + \lambda_2 \overrightarrow{F_2} + \lambda_3 \overrightarrow{F_3},\ \lambda_i \in \mathrm{R}, \mathrm{i} = 1, 2, 3$$

Thus, we can get the equations

$$\vec{X} \cdot \overrightarrow{F_1} = \lambda_1 + \lambda_2 \overrightarrow{F_1} \cdot \overrightarrow{F_2} = \cos\alpha$$

$$\vec{X} \cdot \overrightarrow{F_2} = \lambda_1 \overrightarrow{F_1} \cdot \overrightarrow{F_2} + \lambda_2 = \cos\left(\frac{\pi}{2} + \alpha\right) = -\sin\alpha$$

$$\vec{X} \cdot \overrightarrow{F_3} = \lambda_3 = \cos\theta$$

From these equations simply, we obtain following facts,

$$\lambda_1 + \lambda_2 \cos 2c = \cos\alpha$$
$$\lambda_1 \cos 2c + \lambda_2 = -\sin\alpha$$
$$\lambda_3 = \cos\theta$$

By solving these equations, we obtain easily λ_1, λ_2, and λ_3. That is

$$\lambda_1 = \frac{\begin{vmatrix} \cos\alpha & \cos 2c \\ -\sin\alpha & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \cos 2c \\ \cos 2c & 1 \end{vmatrix}} = \frac{\cos\alpha + \sin\alpha\cos 2c}{\sin^2 2c}$$

$$\lambda_2 = \frac{\begin{vmatrix} 1 & \cos\alpha \\ \cos 2c & -\sin\alpha \end{vmatrix}}{\begin{vmatrix} 1 & \cos 2c \\ \cos 2c & 1 \end{vmatrix}} = \frac{-\sin\alpha - \cos\alpha\cos 2c}{\sin^2 2c}$$

$$\lambda_3 = \cos\theta$$

Thus, for any point X on S^2 we can write

$$\vec{X} = \frac{\cos\alpha + \sin\alpha\cos 2c}{\sin^2 2c}\overrightarrow{F_1} + \frac{-\sin\alpha - \cos\alpha\cos 2c}{\sin^2 2c}\overrightarrow{F_2} + \cos\theta\,\overrightarrow{F_3} \tag{9}$$

and we know

$$\vec{X} = (x_1, x_2, x_3) \text{ and } \|\vec{X}\|^2 = 1$$

After combining (3) and (9), we have $\vec{X}$ depending on only one parameter, that is shown below in Example 4.

Example 4: Let us take $\overrightarrow{F_1} = (1,0,0)$ and $\overrightarrow{F_2} = (0,1,0)$ then $\overrightarrow{F_3} = \frac{\overrightarrow{F_1}\times\overrightarrow{F_2}}{\|\overrightarrow{F_1}\times\overrightarrow{F_2}\|} = (0,0,1)$. If we plug in $\overrightarrow{F_1}$, $\overrightarrow{F_2}$, and $\overrightarrow{F_3}$ in equation (9), we have equation (10), which is

$$\vec{X} = \left(\frac{\cos\alpha + \sin\alpha\cos 2c}{\sin^2 2c}, \frac{-\sin\alpha - \cos\alpha\cos 2c}{\sin^2 2c}, \cos\theta\right) \tag{10}$$

Clearly $d(F_1, F_2) = 2c$,

$$2c = \frac{\pi}{2}$$

and we know

$$\vec{X} = (x_1, x_2, x_3),\ \|\vec{X}\|^2 = 1,\ X \in S^2$$

Using these facts in equation (10), simply we have

$$\left(\frac{\cos\alpha + \sin\alpha\cos 2c}{\sin^2 2c}\right)^2 + \left(\frac{-\sin\alpha - \cos\alpha\cos 2c}{\sin^2 2c}\right)^2 + \cos^2\theta = 1$$

$$\left(\frac{\cos\alpha + \sin\alpha.0}{1}\right)^2 + \left(\frac{-\sin\alpha - \cos\alpha.0}{1}\right)^2 + \cos^2\theta = 1$$

$$1 - \cos^2\alpha - \sin^2\alpha = \cos^2\theta$$

$$0 = \cos\theta$$

So that

$$\vec{X} = (\cos\alpha, -\sin\alpha, 0)$$

Hence, we can get the following parametric equations

$$x_1 = \cos\alpha$$
$$x_2 = -\sin\alpha$$
$$x_3 = 0$$

We can easily see that it's a circle.

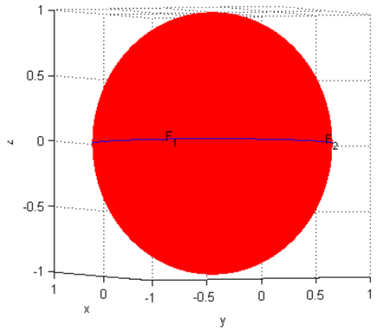

Figure 13: Spherical Parabola (Plot produced with Matlab by using the command below)

```
hold on;ezplot3('cos(t)','-sin(t)','0'),sphere(100),
text(1,0,0,'.F_1','FontSize',10),text(0,1,0,'.F_2','FontSize',10)
```

5. Conclusions

Oriented lines in an Euclidean 3-dimension space R^3 may be represented by unit vectors with three components over the ring of dual numbers. A differentiable curve on the dual unit sphere which depends on a real parameter α corresponds to a ruled surface in R^3. This correspondence is one-to-one and allows the geometry of dual spherical curves on a unit dual sphere. It is known as E.Study Mapping.

Ruled surfaces are one of the most important topics of differential geometry. A ruled surface can always be described as the set of points swept by a moving straight line. These kind of surfaces are used in many areas of sciences such as Computer Aided Geometric Design(CAGD), mathematical physics, moving geometry and kinematics for modelling the problems. So many geometers and engineers have studied on ruled surfaces in different spaces.

Since building materials such as wood are straight, they can be considered as straight lines. The result is that if engineers are planning to construct something with curvature, they can use ruled surface since all the lines are straight [4].

As we said above, a differentiable curve on the dual unit sphere which depends on a real parameter α corresponds to a ruled surface in R^3, by our equations we can apply the E.Study map of the spherical conics and this will be beneficial in the stated areas.

6. Supplementary Electronic Materials

1. Matlab file 1 (spherical_ellipse1).
2. Matlab file 2 (spherical_ellipse2).

7. References

1. Xiong, J. , *Geometry and Singularities of Spatial and Spherical Curves*, Doctoral Thesis, University of Hawai, 2004
2. Maeda, Y. , *Spherical Conics and the Fourth Parameter*, KMITL Sci. J.Vol. 5 No.1. ,2005
3. Namikawa, Y. , *On Spherical Hyperbolas*, (Japanese) Sugaku 11, 1959
4. Orbay K. , Kasap E. , Aydemir I. ,*Mannheim Offsets of Ruled Surfaces*, Mathematical Problems in Engineering, 2009